J. Appl. Maths Mechs Vol. 56, No. 3, pp. 303–318, 1992
Printed in Great Britain.

0021–8928/92 \$15.00 + .00

THE ASYMPTOTIC METHOD IN PROBLEMS ON THE MOTION OF BODIES IN A MELTING MEDIUM†

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(*Received* 25 *April* 1991)

Bodies moving in a melting solid medium or one body sliding over the surface of another, with the formation of a melt layer in the contact zone, are considered in a two-dimensional setting. The asymptotic method proposed in [1] for the case of moving plates is extended to deal with quite general shapes. Among the bodies considered are wedges and transversely moving circular cylinders. A detailed study is made of a beam sliding over a plane surface, with the beam material melting in the contact zone.

CONSIDER a solid medium impinging at a constant velocity V on a fixed contour (Fig. 1); the temperature of the medium far upstream from the contour is T_∞; a melt or vapour layer forms near the contour. We will introduce the coordinates usually employed in thin-layer theory: x along the contour and y along the normal to it; the origin O may be placed at an arbitrary point of the contour. The angle between an element of the contour and the direction of the velocity of the solid medium is denoted by $\gamma(x)$ and the y coordinate of the solid–liquid interface by $y^*(x)$. We will assume that the points of the contour may move along it at a velocity $U(x)$.

The flow and temperature distribution in the melt are assumed to satisfy the usual equations of thin-layer theory:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial y^2},\quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$
$$\rho c\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\lambda\frac{\partial^2 T}{\partial y^2}+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{1}$$

where u and v are the velocity components in the x and y directions, p is the pressure and T is the temperature. The physical parameters of the melt—its density ρ, specific heat c and the coefficient of viscosity and thermal conductivity μ and λ are assumed to be constant. The corresponding quantities ρ_s, c_s and λ_s for the solid medium, as well as the specific latent heat h_f absorbed when it melts and its melting point T_w, are also assumed to be constants.

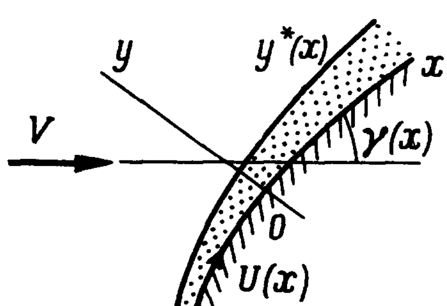

FIG. 1.

† *Prikl. Mat. Mekh.* Vol. 56, No. 3, pp. 368–385, 1992.

In accordance with the equation of continuity, we introduce the stream function ψ, so that $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$.

The boundary conditions for equations (1) are as follows:

$$y=0:\ \psi=0,\quad \partial\psi/\partial y=U(x),\quad T=T_w(x) \tag{2}$$

$$\begin{gathered} y=y^*:\ u=V\cos\gamma-(V\sin\gamma+V)dy^*/dx,\quad T=T_m \\ \lambda\partial T/\partial y\,dx-\rho h_f d\psi=\lambda_s\partial T_s/\partial y\,dx \end{gathered} \tag{3}$$

(the temperature T_w need not be given; it can be determined from other conditions, such as $\partial T/\partial y|_w = 0$).

The last condition of (3) is formulated in the thin-layer approximation; everything else is valid in the exact version of the problem.

The expression $\lambda_s\partial T_s/\partial y$ in the equation of heat balance at the melt surface is determined by finding the temperature distribution in the solid medium, that is, by solving the equation

$$\rho_s V c_s \frac{\partial T}{\partial x} = \lambda_s\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{4}$$

with boundary conditions

$$y=y^*:\ T=T_m,\quad y\to\infty:\ T=T_\infty \tag{5}$$

The need to determine the previously unknown boundary $y = y^*(x)$ links this problem with the problem of flow in a melt layer.

Following an idea in [1], we replace x and y by new independent variables x and $\zeta = y/y^*$, expressing the stream function as $\psi = y^*\varphi(\zeta, x)$ and replacing dp/dx by a new quantity $k(x)$ defined by the equation

$$-\mu^{-1}(dp/dx)y^{*2}=k(x) \tag{6}$$

When the stream function ψ is used, the second equation of system (1) will be satisfied, but the momentum and temperature equations become, in the new variables

$$\begin{gathered} \nu(\varphi''' + k) = y^{*2}\left(\varphi'\frac{\partial\varphi'}{\partial x} - \varphi''\frac{\partial\varphi}{\partial x}\right) - y^*y^{\cdot *}\varphi\varphi'' \\ \lambda T'' + \mu\varphi''^2 = \rho c\left[y^{*2}\left(\varphi'\frac{\partial T}{\partial x} - T'\frac{\partial\varphi}{\partial x}\right) - y^*y^{\cdot *}\varphi T'\right] \end{gathered} \tag{7}$$

The primes on φ and T denote partial derivatives with respect to ζ; $y^{\cdot *} = dy^*/dx$.

The boundary conditions (2) and (3) for the flow in the layer are transformed as follows:

$$\zeta=0:\ \varphi=0,\quad \varphi'=U(x),\quad T=T_w \tag{8a}$$

$$\begin{gathered} \zeta=1:\ \varphi'(1+y^{\cdot *2})=V\cos\gamma+(N-1)V\sin\gamma y^{\cdot *}+NV\cos\gamma y^{\cdot *2},\quad T=T_m, \\ \rho d(y^*\varphi)=\rho_s V(\sin\gamma dx+\cos\gamma dy^*) \\ \lambda T'dx+\rho h_f y^* d(y^*\varphi)=\lambda_s y^*(\partial T_s/\partial y)dx \end{gathered} \tag{8b}$$

The condition for $\varphi'(1)$ is obtained by combining the first and third conditions of (3).

To apply our asymptotic method, we will assume that y^* and its derivative $y^{\cdot *}$ are small. Conditions ensuring that y^* is indeed small compared with the characteristic longitudinal scale of the liquid layer will be determined as the problem is being solved.

We will represent φ, T, k and y^* in the form of infinite sums:

$$\begin{gathered} \varphi=\varphi_0+\varphi_1+\varphi_2+\ldots,\quad T=T_0+T_1+T_2+\ldots \\ k=k_0+k_1+k_2+\ldots,\quad y^*=y_1+y_2+y_3+\ldots \end{gathered} \tag{9}$$

where the subscript of each term indicates its order of smallness. These representations are not unique: each term of the sum may also include part of a higher-order expression. Substituting (9) into Eqs (7) and the boundary conditions (8), adding terms of the same order and equating the sums to zero, we obtain systems of equations for the successive terms in (9). We will present these systems for the quantities written out explicitly in (9).

By equations (7),

$$
\begin{gathered}
\varphi_0''' + k_0 = 0, \quad \lambda T_0'' + \mu \varphi_0''^2 = 0 \\
\varphi_1''' + k_1 = 0, \quad \lambda T_1'' + 2\mu \varphi_0'' \varphi_1'' = 0 \\
\nu (\varphi_2''' + k_2) = y_1^2 \left(\varphi_0' \frac{\partial \varphi_0'}{\partial x} - \varphi_0'' \frac{\partial \varphi_0}{\partial x} \right) - y_1 y_1^{\cdot} \varphi_0 \varphi_0'' \\
\lambda T_2'' + \mu (\varphi_1''^2 + 2\varphi_0'' \varphi_2'') = \rho c \left[y_1^2 \left(\varphi_0' \frac{\partial T_0}{\partial x} - T_0' \frac{\partial \varphi_0}{\partial x} \right) - y_1 y_1^{\cdot} \varphi_0 T_0' \right]
\end{gathered}
\tag{10}
$$

The boundary conditions (8) give

$$
\zeta = 0: \ \varphi_0 = 0, \quad \varphi_0' = U, \quad T_0 = T_w \tag{11}
$$

$$
\begin{gathered}
\varphi_1 = 0, \quad \varphi_1' = 0, \quad T_1 = 0, \quad \varphi_2 = 0, \quad \varphi_2' = 0, \quad T_2 = 0 \\
\zeta = 1: \ \varphi_0' = V \cos \gamma, \quad T_0 = T_m, \quad \varphi_1' = (N-1) V \sin \gamma y_1^{\cdot}, \quad T_1 = 0 \\
\varphi_2' = (NV \cos \gamma - \varphi_0') y_1^{\cdot 2} + (N-1) \sin \gamma y_2^{\cdot}, \quad T_2 = 0 \\
(\varphi_0 - NV \cos \gamma) y_1^{\cdot} + \frac{\partial \varphi_0}{\partial x} y_1 = NV \sin \gamma \\
(\varphi_0 - NV \cos \gamma) y_2^{\cdot} + \frac{\partial \varphi_0}{\partial x} y_2 = -\left(\varphi_1 y_1^{\cdot} + \frac{\partial \varphi_1}{\partial x} y_1 \right) \\
(\varphi_0 - NV \cos \gamma) y_3^{\cdot} + \frac{\partial \varphi_0}{\partial x} y_3 = -\frac{\partial}{\partial x} (\varphi_1 y_2 + \varphi_2 y_1) \\
\lambda T_0' + \rho_s V h_f (y_1 \sin \gamma + y_1 y_1^{\cdot} \cos \gamma) = \lambda_s y_1 \frac{\partial T_s}{\partial y} \\
\lambda T_1' + \rho_s V h_f [y_2 \sin \gamma + (y_1 y_2^{\cdot} + y_2 y_1^{\cdot}) \cos \gamma] = \lambda_s y_2 \frac{\partial T_s}{\partial y} \\
\lambda T_2' + \rho_s V h_f [y_3 \sin \gamma + (y_1 y_3^{\cdot} + y_2 y_2^{\cdot} + y_3 y_1^{\cdot}) \cos \gamma] = \lambda_s y_3 \frac{\partial T_s}{\partial y}
\end{gathered}
\tag{12}
$$

Since the balance equations at the solid–liquid interface [the third and fourth equations of (8b) with $\zeta = 1$] determine the order of y^* and its actual magnitude, we must retain the principal parts of all the terms in the principal approximations of Eqs (12) that determine y.

It should be noted that the successive terms in expressions (9) for φ and T are polynomials of increasing order in ζ, with coefficients depending on x. Convective transfer is manifested in these expressions only in terms of second and higher orders.

By the first equation of (10), φ_0 is a polynomial of the third degree in ζ; determining its coefficients by using the first two conditions of (11) and the first condition of (12), we find

$$
\begin{gathered}
\varphi_0 = U\zeta + \frac{1}{2} \left(V \cos \gamma - U + \frac{k_0}{2} \right) \zeta^2 - \frac{k_0}{6} \zeta^3 \\
\varphi_0' = U + \left(V \cos \gamma - U + \frac{k_0}{2} \right) \zeta - \frac{k_0}{2} \zeta^2
\end{gathered}
\tag{13}
$$

By the second equation of (10), the temperature T_0 is linear in ζ if viscous dissipation is ignored and a fourth-degree polynomial otherwise. Determining the coefficients by using the third condition of (11) and the second condition of (12), we find

$$T_0 = T_w - (T_w - T_m)\zeta + \frac{\mu}{\lambda}\zeta(1-\zeta)\Big[\frac{\alpha_0^2}{2} - \frac{\alpha_0 k_0}{3} + \frac{k_0^2}{12} + \left(-\frac{\alpha_0 k_0}{3} + \frac{k_0^2}{12}\right)\zeta + \frac{k_0^2}{12}\zeta^2\Big]$$

$$T_0' = -(T_w - T_m) + \frac{\mu}{\lambda}\left(\frac{\alpha_0^2}{2} - \frac{\alpha_0 k_0}{3} - \frac{k_0^2}{12} - \alpha_0^2\zeta + \alpha_0 k_0 \zeta^2 - \frac{k_0^2}{3}\zeta^3\right) \qquad (14)$$

$$\left(\alpha_0 = \varphi_0''(0, x)_\zeta^* = V\cos\gamma - U + \frac{k_0}{2}\right)$$

The two conditions of (12) specifying the values of φ_0 and T_0 at $\zeta = 1$, which represent conditions of mass balance and heat flux balance at the solid–liquid interface, are now treated by substituting $\varphi_0(1, x)$, $\partial\varphi_0/\partial x|_{\zeta=1}$ and $T_0'(1, x)$ into them from (13) and (14); this gives a system of two ordinary first-order differential equations for $y^*(x)$and $k_0(x)$:

$$\left[U - (1-2N)V\cos\gamma + \frac{k_0}{6}\right]y^{*\prime} + \left(\frac{dU}{dx} + V\frac{d\cos\gamma}{dx} + \frac{1}{6}\frac{dk_0}{dx}\right)y^* = 2NV\sin\gamma$$

$$-\lambda T_0'(1) = \lambda(T_w - T_m) + \mu\left(\frac{\alpha_0^2}{2} - \frac{2}{3}\alpha_0 k_0 + \frac{k_0^2}{4}\right) = \rho_s V h_f y^*(\sin\gamma + \cos\gamma y^{*\prime}) - \lambda_s y^* \frac{\partial T_s}{\partial y} \qquad (15)$$

To determine the general solution of these equations, taking account of (6), we need three boundary conditions for $y^*(x)$, $k(x)$ and $p(x)$.

Note that if viscous dissipation in the layer is ignored, the term including μ in the heat flux balance equation drops out and $y^*(x)$ can be determined from that equation, independently of the mass balance equation. The latter will then be used to determine $k_0(x)$.

We will first consider the problem of melt flow in the layer at a straight portion of the contour. In that case $\gamma =$ const and the initial mass balance equation (8) at the melt surface, $\zeta = 1$, has a closed integral (henceforth we shall omit the subscript 0 denoting the principal approximation):

$$\rho y^*\varphi = \rho_s V[(x - x_0)\sin\gamma + (y^* - y_0^*)\cos\gamma] \qquad (16)$$

The constant of integration is so chosen that the discharge of melt in the cross-section at x_0 with the appropriate value of $y^* = y_0^*$ is zero (if $y_0^* \neq 0$ the melt will flow away from the cross-section $x = x_0$ in both directions).

Substituting $\varphi(1)$ into (16), we get an integral of system (15):

$$k = 6(2N-1)V\cos\gamma - 6U + 12NV[(x - x_0)\sin\gamma - y_0^*\cos\gamma]/y^* \qquad (17)$$

connecting the functions $k(x)$ and $y^*(x)$.

We shall ignore the heat flux into the solid medium in the heat-balance equation (15). (This flux will vanish if the medium is heated up to its melting point, that is, $T_\infty = T_w$; approximate allowance may be made for the heat flux by replacing h_f by an "effective" latent heat $h_f + c_s(T_m - T_\infty)$.) Then the equation, with α replaced by the appropriate expression and the integral (17) suitably inserted, will determine $y^*(x)$:

$$-\lambda T'(1) = \lambda(T_w - T_m) + \mu\left[\frac{1}{2}(V\cos\gamma - U)^2 - \frac{k}{6}\left(V\cos\gamma - U\right) + \frac{k^2}{24}\right] = \rho_s V h_f\left(\sin\gamma + \cos\gamma\frac{dy^*}{dx}\right)y^* \qquad (18)$$

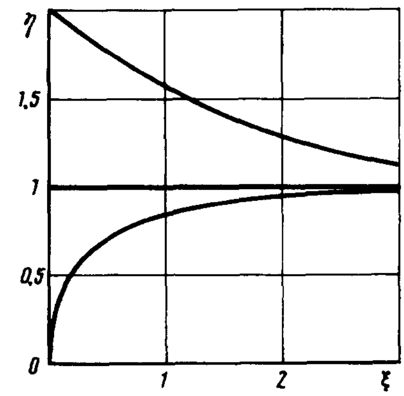

FIG. 2.

Let us assume that dissipation in the layer may be ignored. As already stated, this means that equation (18) becomes independent of (17), and if γ = const it is easily integrated.

If $\gamma = \pi/2$ the equation degenerates into a finite relation

$$y^* = \frac{\lambda(T_w - T_m)}{\rho_s V h_f} \tag{19}$$

and we need only two boundary conditions to determine the constants of integration of systems (17), (18) and (6), rather than three as in the general case.

The $\gamma = 0$, assuming that the wall temperature T_w is constant, we obtain (the generalization to the case $T_w \neq$ const is obvious)

$$y^{*2} = y_0^{*2} + 2\frac{\lambda(T_w - T_m)}{\rho_s V h_f}(x - x_0) \tag{20}$$

If $\gamma \neq 0$, $\pi/2$, the solution for T_w = const is expressed by the formulas

$$y^* = \frac{\lambda(T_w - T_m)}{\rho_s V h_f \sin\gamma} = y_\infty^* = \text{const} \tag{21}$$

$$x - x_0 = -\operatorname{ctg}\gamma\left(y^* - y_0^* + y_\infty^* \ln\left|\frac{y_\infty^* - y^*}{y_\infty^* - y_0^*}\right|\right)$$

Figure 2 shows plots of this solution, apart from a translation in the x direction, in variables $\xi = x\operatorname{tg}\gamma/y_\infty^*$, $\eta = y^*/y_\infty^*$. As x increases the layer thickness increases as long as $y_0^* < y_\infty^*$ and decreases when $y_0^* > y_\infty^*$; at $y_0^* = y_\infty^*$ it remains constant. In the first case the initial thickness of the layer may, in particular, be zero.

Let us study the behaviour of the other flow parameters in the layer, between the limits $x = 0$ and $x = L$, for the simplest case: y^* = const. To simplify the formulas, we take $\gamma = \pi/2$ (the generalization to the case $\gamma \neq 0$, $\pi/2$ presents no difficulties). Then the integral (17) becomes

$$k = -6U + 12NV(x - x_0)/y^*$$

Integrating (6) with respect to x and taking the pressure at $x = L$ as the initial pressure level, we obtain

$$\frac{y^{*2}}{\mu}p = \left(6U + 12NV\frac{x_0}{y^*}\right)(x - L) - \frac{6NV}{y^*}(x^2 - L^2) \tag{22}$$

Integrating p again with respect to x, from 0 to L, we find an expression for the total normal force acting on the layer over the interval $(0, L)$:

$$Y = \frac{4\mu NVL^3}{y^{*3}}\left(1 - \frac{3}{2}\frac{x_0}{L} - \frac{3}{4}\beta\right); \quad \beta = \frac{y^*U}{NVL}$$

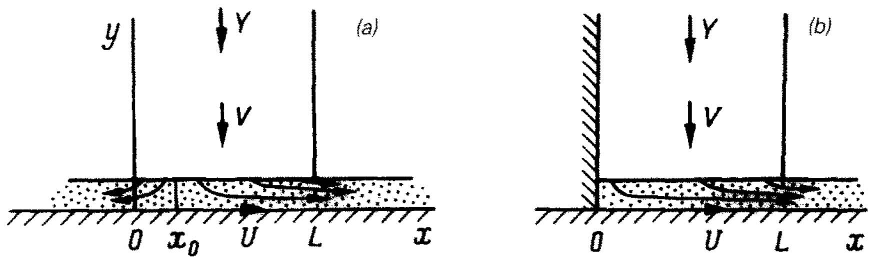

FIG. 3.

Here β is a parameter representing the influence of the velocity U of the surface.

As to the distribution of frictional stress on the plate, τ_w, and at the melting surface of the solid medium, τ^*, we obtain

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_w = \frac{\mu}{y^*}\varphi''(0) = \frac{\mu U}{y^*}\left(-4 + \frac{6}{\beta}\frac{x - x_0}{L}\right)$$

$$\tau^* = -\mu \frac{\partial u}{\partial y}\bigg|^* = \frac{\mu}{y^*}\varphi''(1) = \frac{\mu U}{y^*}\left(-2 + \frac{6}{\beta}\frac{x - x_0}{L}\right)$$

Hence the total forces of friction on the parts of the plate and melt surface from $x = 0$ to $x = L$, this implies formulas

$$X_w = \frac{\mu U L}{y^*}\left[-4 + \frac{3}{\beta}\left(1 - 2\frac{x_0}{L}\right)\right], \quad X^* = \frac{\mu U L}{y^*}\left[-2 + \frac{3}{\beta}\left(1 - 2\frac{x_0}{L}\right)\right]$$

Let us consider the application of the formulas developed above to a few specific problems.

Consider a beam of constant finite width L, pressed by a normally applied force Y onto a plane moving with velocity U (Fig. 3a, b). On contact with the plane the material of the beam melts and the liquid melt is squeezed out from under the beam, the latter thus approaching the plane at a rate V.

This problem has various applications and has frequently been considered in the literature. The case $U = 0$ was studied in [2]; the authors use the same equation of motion as we have here, but write the heat balance equation for the layer in integral form, with the temperature distribution represented by a quadratic polynomial. The method of integral relations with quadratic polynomials for the velocity and the temperature was used in [3] to examine the general case $U = \text{const} > 0$ and $\mu \geq 0$. Finally, the problem was recently considered with the same equations as in this paper [4].

To apply the above formulas to this problem, we determine x_0, that is, the position of the cross-section at which the discharge of melt is zero. This parameter is determined by the condition that the pressure in the section $x = 0$ is the same as in $x = L$, i.e. zero. Putting $x = 0$ and $p = 0$ in the expression (22) for p, we obtain $x_0 L = \frac{1}{2}(1 - \beta)$. Clearly, x_0/L is the fraction Q of the melt squeezed out from beneath the beam toward the direction of the moving beam. If $Q = 0$ all the melt extruded to the left returns to beneath the beam; if $Q > 0$ there should be a discharge of melt in front of the beam; while the solution for $Q < 0$ corresponds to melt already present on the plate flowing into the space under the beam. Substituting the resulting expression for x_0 into the formulas for p, τ_w and τ^*, we obtain

$$p = \frac{6\mu N V}{y^{*3}} x(L - x)$$

$$\tau_w = \frac{\mu U}{y^*}\left(3\frac{2x - L}{U y^*} N V - 1\right), \quad \tau^* = \frac{\mu U}{y^*}\left(3\frac{2x - L}{U y^*} N V + 1\right)$$

Hence it follows that the pressure distribution is in this case symmetrical about the middle line of the beam and independent of the velocity U. For the total forces Y, X_w and X^*, we have

$$Y = \frac{\mu N V L^3}{y^{*3}}, \quad X^* = -X_w = \frac{\mu U L}{y^*}$$

Note that the friction force X^* acting between the beam and the surface in the presence of the melt layer is the same as the similar force for an ordinary layer of lubricant of the same thickness y^* with Couette flow.

Substituting $y^* = \lambda(T_w = T_m)/(\rho_s V h_f)$, into these formulas, we obtain

$$Y = N\mu V\left[\frac{\rho_s V L h_f}{\lambda(T_w - T_m)}\right]^3, \quad X^3 = -X_w = \frac{\mu\rho_s U V L^3}{\lambda(T_w - T_m)}$$

The first of these formulas establishes the relation between the force Y pressing down the beam and the rate of melting V: the rate of melting is proportional to $Y^{1/4}$. The formula for $\overline{Q}$ yields a relationship between this quantity and the pressing force Y:

$$\overline{Q} = \frac{1}{2} - \left(\frac{\rho U L h_f}{\lambda(T_w - T_m)}\right)^{1/4}\left(\frac{\mu U}{Y}\right)^{1/4}$$

If it is assumed that there is initially no liquid layer on the plate, then necessarily $Q \geqslant 0$. The minimum value of the pressing force for which $Q = 0$ and all the melted material is pushed out from under the beam by the moving plate is

$$Y_{\min} = 4\mu U \frac{\rho U L h_f}{\lambda(T_w - T_m)}$$

As Y increases above this value, an increasingly larger part of the melt is forced out from under the beam in the direction of motion of the plate; if Y increases without limit this part tends to a half of the total discharge of melt.

It should further be noted that the absolute values of the forces X_w and X^* are indeed equal only in the principal approximation. In reality, taking into account that $p(0) = p(L)$, the sum of these two forces must equal the difference between the projections onto the x direction of the momenta of the liquid in the layer at $x = L$ and $x = 0$; this difference is readily seen to be $\rho_s VLU$. This quantity determines the order of accuracy of the expressions for X_w and X^*.

We can also write down an averaged coefficient of friction between the beam and the moving surface in the presence of a melt layer:

$$\mu_f = \frac{X^*}{Y} = \left(\frac{\mu U}{Y}\right)^{1/4}\left(\frac{\rho U L h_f}{\lambda(T_w - T_m)}\right)^{3/4}$$

Let us suppose now that the beam, while being pressed onto the moving surface, is in contact on the left with a fixed wall, which is impermeable to the melt (Fig. 3b). Then instead of $p(0) = 0$ we must assume that $x_0 = 0$. The required exact condition $u = \varphi'(\zeta)$ at $x = 0$, $0 < \zeta \leqslant 1$, is of course satisfied only on the average over the layer thickness. The general formulas yield, after putting $x_0 = 0$,

$$p = \frac{6\mu N V L^2}{y^{*3}}\left[1 - \frac{x^2}{L^2} - \beta\left(1 - \frac{x}{L}\right)\right] \tag{23}$$

$$\tau_w = \frac{\mu N V L}{y^{*2}}\left(-4\beta + \frac{6x}{L}\right), \quad \tau^* = \frac{\mu N V L}{y^{*2}}\left(-2\beta + \frac{6x}{L}\right)$$

and accordingly

$$Y = \frac{\mu N V L^3}{y^{*3}}(4 - 3\beta)$$

$$X_w = \frac{\mu N V L^2}{y^{*2}}(-4\beta + 3), \quad X^* = \frac{\mu N V L^2}{y^{*2}}(-2\beta + 3) \tag{24}$$

Figure 4 shows a plot of the pressure distribution in dimensionless variables

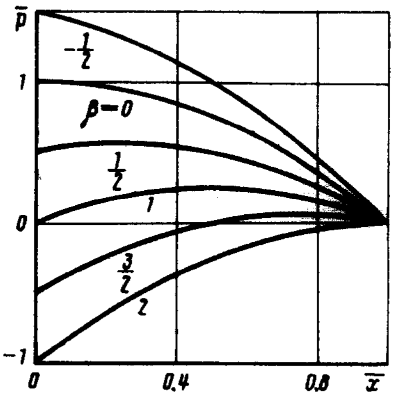

FIG. 4.

$$\bar{p}=\frac{y^{*3}}{6\mu NVL^2}p, \quad \bar{x}=\frac{x}{L}$$

for several values of β. As β is increased, i.e. the velocity U increases, the pressure under the beam drops because of the entraining motion of the moving surface; when $\beta = 1$ the pressure is zero at the left edge of the beam, but if $\beta \geqslant 2$ it falls below the external pressure at every point.

According to (24), the total normal force Y at $\beta \geqslant 4/3$ changes sign and becomes negative, i.e. the beam is pressed onto the heated surface because of the low pressure region formed beneath it.

Substituting the expressions for β and y^* into the formula (24) for Y we find, after reduction, the relationship between Y and the rate of melting V:

$$Y=\frac{\mu U}{N^2}\left(\frac{U}{\alpha}\right)^2\left[4\left(\frac{V}{\alpha}\right)^4-3\left(\frac{V}{\alpha}\right)^2\right] \quad \left(\alpha=\left(\frac{\lambda\Delta TU}{N\rho_s h_f L}\right)^{1/2}\right).$$

A plot of this relationship in variables

$$\bar{Y}=\frac{N^2}{\mu U}\left(\frac{\alpha}{U}\right)^2 Y, \quad \frac{V}{\alpha}$$

is shown in Fig. 5; as the load is reduced, a solution exists down to values

$$Y_{\min}=-\frac{9}{16}\mu U\frac{\rho ULh_f}{\lambda\Delta T}$$

The problem as formulated has no solution for smaller Y values. Whtin the range $Y_{\min}<Y<0$ there are two values of the rate of melting for each value of the force. However, the solution for

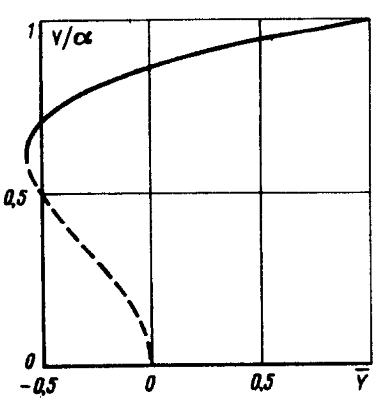

FIG. 5.

smaller V (the dashed portion of the curve in Fig. 5) must be discarded, since it implies that as the load is increased the layer thickness y^*, which is inversely proportional to V, also increases; while one would expect it to decrease.

Looking at formulas (24) for X_w and X^*, we see that if the force Y is fixed and the velocity U increases, the friction at the wall X_w will change sign at $\beta = 3/4$; instead of pulling, it begins to exert a retarding effect. The same happens to the force of friction X^* at the beam, but at $\beta = 3/2$. This change is easily explained by the behaviour of the velocity profile of the melt at different sections of the layer, which is due in turn to the changes in the pressure distribution as β varies.

We will now consider symmetric motion of a wedge "melting its way" through a medium (this problem was considered previously [5] by the method of integral relations). The origin of the system of coordinates is placed at the tip of the wedge, and it is assumed that the initial thickness of the melt layer at the tip is zero, i.e. we are assuming that the distance L_T of the melting front ahead of the tip is of the order of $\lambda(T_w - T_m)(\rho_s V h_f)^{-1}$, which is negligibly small compared with the length L of the wedge wall (see [1]).

The shape of the melting front is described by the second expression of (21) with $y_0^* = 0$ and is represented by the lower curve in Fig. 2.

As x increases without limit, the thickness of the melt layer will tend to a constant y_∞^*; hence the rate of melting must also increase without limit. It can be shown that as $x \to \infty$,

$$u_{\max} \to {}^3/_2 NV \sin \gamma x$$

Using expression (17) for k_0 with $x_0 = y_0^* = 0$ and $U = 0$, as well as the relationship between k_0 and dp/dx, again with $p = 0$ at $x = L$, we obtain the following formula for the pressure distribution p:

$$p = \frac{\rho_s V^2 h_f \mu \cos^2 \gamma}{\lambda (T_w - T_m)} [\bar{p}(\xi) - \bar{p}(\xi_L)]$$

$$\left(\bar{p}(\xi) = \int_\xi^1 \frac{6(2N-1)\eta + 12N\xi}{\eta^3} d\xi, \quad \xi_L = \frac{L \operatorname{tg} \gamma}{y_\infty^*}\right)$$

Figure 6 plots the function $\bar{p}(\xi)$ for several N values. It is readily seen that at $\xi = 0$ and $N = 1/2$, the pressure has a logarithmic singularity ($\bar{p} \to 6(2N-1)\ln\xi$), while at large ξ values (when $\eta \to 1$) $\bar{p} \to -\infty$ as $-6N\xi^2$.

Integrating (25), we obtain the total normal pressure force acting on a length L of the side of the wedge:

$$P = \mu V \frac{\cos^3 \gamma}{\sin \gamma} \left[\int_0^{\xi_L} \bar{p}(\xi)\, d\xi - \xi_L \bar{p}(\xi_L) \right] = \mu V \frac{\cos^3 \gamma}{\sin \gamma} \bar{P}$$

The frictional force acting along a length L of the side of the wedge is

$$T_w = \mu \int_0^L \frac{\partial u}{\partial y}\bigg|_w dx = \mu \int_0^L \frac{\varphi''(0, x)}{y^*} dx$$

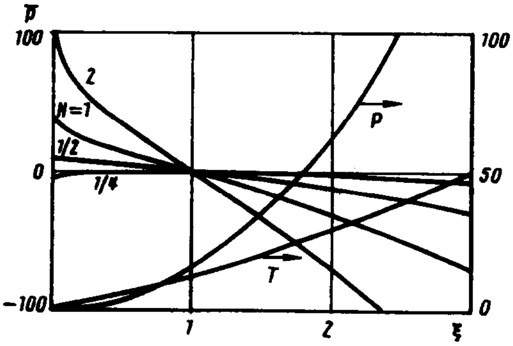

FIG. 6.

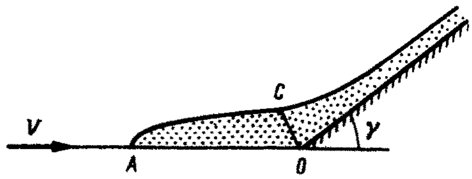

FIG. 7.

In the approximation adopted above,

$$T_w = \mu \int \frac{\alpha_0(x)}{y^*} dx = \frac{\mu V \cos^2\gamma}{\sin\gamma} \int_0^{\xi_L} \left[\frac{6N-2}{\eta} + 6N \frac{\xi}{\eta^2} \right] d\xi = \frac{\mu V \cos^2\gamma}{\sin\gamma} \bar{T}$$

The quantities $\bar{p}$ and $\bar{T}$ are plotted against ξ_L, for $N = 1$, in Fig. 6.

We will now consider the qualitative behaviour of the melt layer in the configuration shown in Fig. 7: upstream of the wedge lies a heated length of wall, AO parallel to the direction of the moving solid medium. Since the thickness of the melt layer increases along this wall [see (20)] as the square root of the distance from its initial point, the layer thickness $y_0{}^*$ at the angle in the wall may be smaller or greater than the limiting thickness $y_\infty{}^*$ of the layer over the wedge. Whether the thickness will increase or decrease beyond that point will depend on the second relation in formula (21) for y^*.

Of course, an examination of the melt flow near the angle in the wall no longer falls within the range of applicability of thin-layer theory. Nevertheless, as an approximation, the solutions for the two parts of the layer that meet at the angle may be combined on the assumption that the melt discharge, layer thickness and pressure are continuous across the cross-section OC (see Fig. 7, which corresponds to the case $y_0{}^* > y_\infty{}^*$).

It is more difficult to investigate flows allowing for the conduction of heat due to viscous dissipation. We again consider the problem of the formation of a melt layer between a moving plane (plate) and a melting medium pressed onto it. Two cases will be considered separately: a thermally insulated plate, when all the heat necessary to melt the medium is generated by viscous dissipation in the layer; and a plate of given temperature.

In the first case it follows from the second expression of (14), via the condition $T_0'(0) = 0$, that

$$T_w = T_m + \frac{\mu}{12\lambda}\left(6U^2 - 2Uk_0 + \frac{k_0^2}{2}\right), \quad T'(1) = -\frac{\mu}{\lambda}\left(\frac{k_0^2}{12} + U^2\right)$$

The conditions of mass balance (17) and heat balance (18) are

$$k_0 = -6U + 12NV\frac{x - x_0}{y^*} \tag{25}$$

$$-\mu\left(\frac{k_0^2}{12} + U^2\right) + \rho_s V h_f y^* = 0 \tag{26}$$

Thus, y^* is expressed as a function of $x - x_0$ parametrically, in terms of k_0; y^* is a quadratic polynomial and $x - x_0$ is a cubic polynomial in k_0.

Introducing new variables

$$x = \frac{\mu U^3}{12N\rho_s V^2 h_f}\bar{x}, \quad y^* = \frac{\mu U^2}{\rho_g V h_f}\bar{y}, \quad k_0 = U\bar{k}, \quad p = \frac{\rho h_f}{12}\bar{p} \tag{27}$$

we express these functions as

$$\bar{y} = 1 + \frac{\bar{k}^2}{12}, \quad \bar{x} - \bar{x}_0 = (\bar{k} + 6)\left(1 + \frac{\bar{k}^2}{12}\right) = \frac{16}{3} + \frac{(\bar{k}+2)^3}{12}$$

The pressure distribution (apart from a constant) and wall temperature are

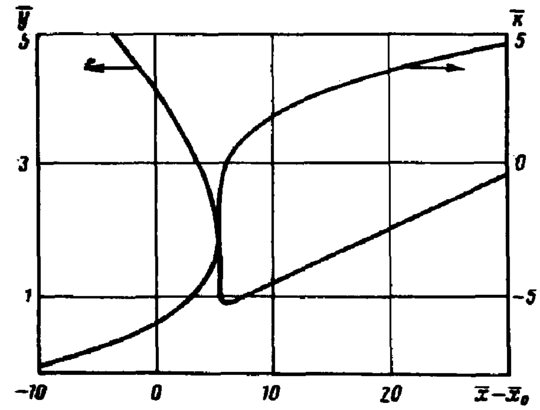

FIG. 8.

$$\bar{p} = -\int_0^{\bar{x}-\bar{x}_0} \frac{\bar{k}}{\bar{y}^2}\, d(\bar{x}-\bar{x}_0), \quad \Delta\bar{T} = \frac{12\lambda (T_w - T_m)}{\mu U^2} = \frac{\bar{k}^2}{2} - 2\bar{k} + 6$$

Figure 8 illustrates the behaviour of the melt layer thickness $\bar{y}$ and pressure gradient $\bar{k}$ in dimensionless variables. The layer thickness first decreases, falling at $\bar{x} = \bar{x}_0 + 6$ to a minimum value $y^*_{\min} = \mu U^2/(\rho_s V h_f)$, and then increases again. The quantity $\bar{k}$ increases monotonically as $\bar{x}$ increases and vanishes at $\bar{x} = \bar{x}_0 + 6$, i.e. where the layer thickness is a minimum; the pressure reaches a maximum in that cross-section.

Plots of pressure and temperature, in dimensionless variables, are given in Fig. 9.

The wall temperature first falls monotonically, reaching a minimum below the cross-section with the least layer thickness, and then increases again. We note that at $\bar{x} - \bar{x}_0 = 16/3$ the derivative $\partial T/\partial x|_w$ becomes infinite, so that, strictly speaking, the underlying assumption of thin-layer theory—the possibility of ignoring the heat flux in the longitudinal direction—fails near this cross-section.

We now again apply the universal functions obtained above, to investigate melting of a beam of width L pressed onto a plate by a force Y. As before, we will have to determine the value of the parameter x_0. To find the dependence of this parameter on the magnitude of the compressive force Y, as well as the rate of melting V, we have two equations:

$$\bar{p}(-\bar{x}_0) = \bar{p}(\bar{x}_L - \bar{x}_0), \quad \frac{12N}{\rho_s h_f L} Y = \frac{1}{\bar{x}_L} \int_{-\bar{x}_0}^{\bar{x}_L - \bar{x}_0} [\bar{p}(\xi) - \bar{p}(-\bar{x}_0)]\, d\xi$$

$$(\bar{x}_L = 12N\rho_s h_f L V^2 (\mu U^3)^{-1})$$

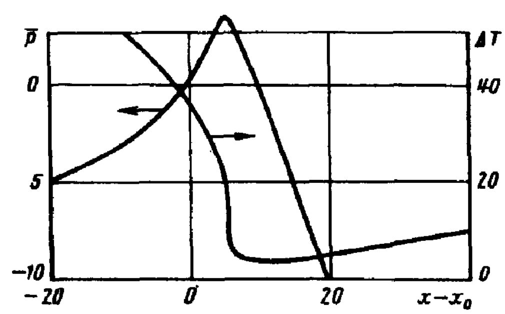

FIG. 9.

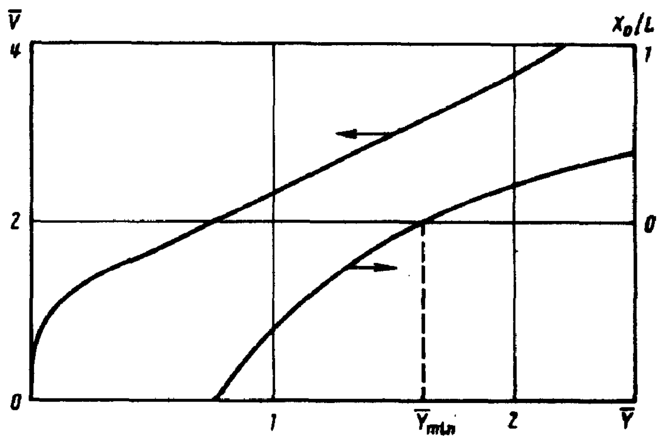

FIG. 10.

Figure 10 illustrates the computed functions

$$\bar{x}_L^{1/3} = (12N\rho_s h_f L(\mu U^3)^{-1})^{1/3} V \text{ and } x_0/L \text{ versus } \bar Y = 12N(\rho_s h_f L)^{-1} Y$$

As an example of the use of these functions, Fig. 11 shows plots of the pressure

$$p = \frac{\rho_s h_f}{12N}\left[\bar p(\bar x - \bar x_0) - \bar p(-\bar x_0)\right]$$

and temperature for several $\bar Y$ values; the open circles on each curve correspond to the cross-sections $x = x_0$.

Suppose now that the temperature of the plate is constant and given at T_w ($> T_m$). Then the mass balance condition [the second equation in (25)] at the solid–liquid interface remains the same, but the heat balance condition [the second relationship of (25)], by (18), must be replaced as follows:

$$\lambda(T_w - T_m) + \mu\left(\frac{k^2}{24} + \frac{Uk}{6} + \frac{1}{2}U^2\right) = \rho_s V h_f y^*$$

Introducing the same dimensionless variables (27), we reduce the defining relations to the form

$$\bar y = \chi + \left(\frac{\bar k^2}{24} + \frac{\bar k}{6} + \frac{1}{2}\right), \quad \bar x - \bar x_0 = (\bar k + 6)\bar y, \quad \chi = \frac{\lambda(T_w - T_m)}{\mu U^2}$$

Thus, unlike the previous case, the defining relations involve a prescribed parameter χ.

Figure 12(a) plots y^* against $x - x_0$ for several χ values. The layer thickness decreases as ξ increases, down to a value $\bar y_{min} = \chi + 1/3$ and $\xi = 4(\chi + 1/3)$, and then again increases.

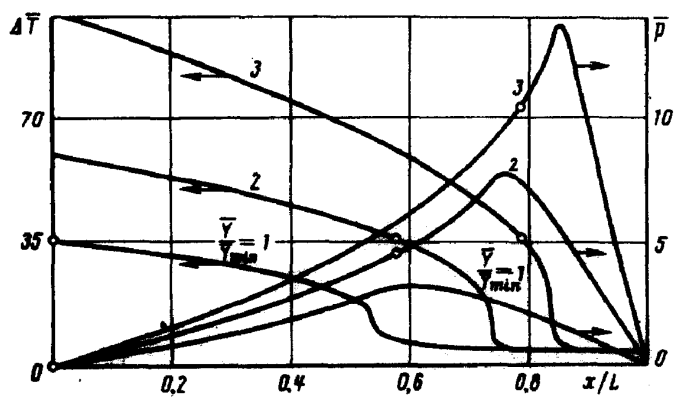

FIG. 11.

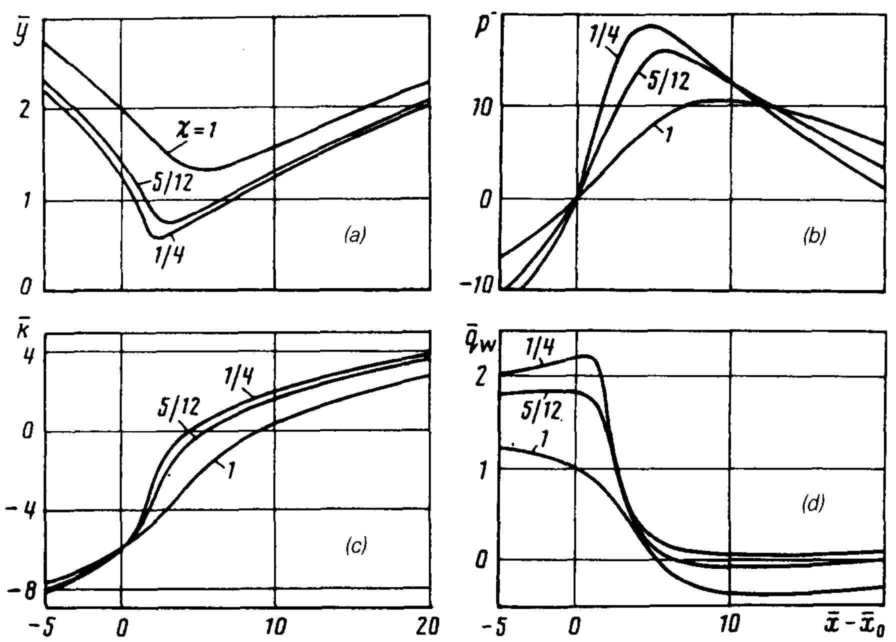

FIG. 12.

The quantity $\bar{k}$ increases with ξ, passing through zero at $\xi = 6(\chi + \frac{1}{2})$. Plots of $\bar{k}(\xi)$ are shown in Fig. 12(b). Accordingly, the pressure first increases, reaching a maximum at $\xi = 6(\chi + \frac{1}{2})$, and then again decreases; pressure curves are shown in Fig. 12(c).

The heat flux to the plate is given by

$$\bar{q}_w = \frac{\lambda}{\rho_s V h_f} \frac{\partial T}{\partial y}\bigg|_w = \frac{1}{\bar{y}}\left(-\chi + \frac{\bar{k}^2}{24} - \frac{\bar{k}}{6} + \frac{1}{2}\right)$$

Plots of this function for three characteristic values of χ, in the ranges $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$, are shown in Fig. 12(d). It is clear that, if the difference $T_w - T_m (\chi < \frac{1}{3})$ is not too large, the heat flux will be directed toward the plate (positive), i.e. the material will melt owing to heat generated by the dissipation of mechnical energy in the melt layer. As $T_w - T_m$ increases, an increasingly large part of the plate will supply heat to the flow, and at a certain point the whole surface beneath the beam will become a source of heat.

To determine the rate of melting V and the fraction of the melt forced out from beneath the beam in different directions, i.e. the quantity x_0, as functions of the force exerted on the beam, one uses the same formulas as for a thermally insulated plate.

In conclusion, we consider the melt layer formed when a circular cylinder moves at right angles to its generator in a melting solid medium. The spherical analogue of this problem was recently considered in the Stokes approximation, ignoring viscous dissipation [5].

In this case (see Fig. 13), $\gamma = \pi/2 - x/R$, where R is the radius of the cylinder and x is the distance along the circumference from the leading point. Formulas (15) then become (again neglecting heat transferred to the solid medium)

$$\left[(1-2N) V \sin\frac{x}{R} + \frac{1}{6}k_0\right] y'^{*} + \left(\frac{V}{R}\cos\frac{x}{R} + \frac{1}{6}\frac{dk_0}{dx}\right) y^{*} = 2NV\cos\frac{x}{R}$$

$$\lambda(T_w - T_m) + \mu\left(\frac{\alpha_0^2}{2} - \frac{2}{3}\alpha_0 k_0 + \frac{1}{4}k_0^2\right) = \rho_s V h_f y^{*}\left(\cos\frac{x}{R} + y'^{*}\sin\frac{x}{R}\right) \quad (28)$$

$$(\alpha_0 = V\sin(x/R) + k_0/2).$$

The initial conditions for the solution of this system will be

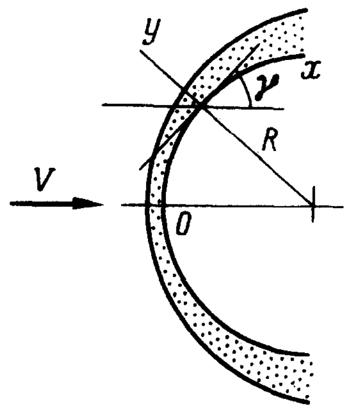

FIG. 13.

$$x=0:\ y^*=y_0=\lambda(T_w-T_m)/(\rho_s V h_f),\ k=0 \tag{29}$$

We introduce dimensionless variables, indicated by bars:

$$x=R\bar{x},\ y^*=R\bar{y},\ k=V\bar{k},\ \alpha=V\bar{\alpha}\ \ (\bar{\alpha}=\sin\bar{x}+\bar{k}/2)$$

System (28) with the initial conditions (29) becomes

$$\begin{gathered}\left[(1-2N)\sin\bar{x}+\frac{1}{6}\bar{k}\right]\frac{d\bar{y}}{d\bar{x}}+\left(\cos\bar{x}+\frac{1}{6}\frac{d\bar{k}}{d\bar{x}}\right)\bar{y}=2N\cos\bar{x}\\ \frac{y_0}{R}+\bar{\mu}\left(\frac{\bar{\alpha}^2}{2}-\frac{2}{3}\bar{\alpha}\bar{k}+\frac{1}{4}\bar{k}^2\right)=\left(\cos\bar{x}+\sin\bar{x}\frac{d\bar{y}}{d\bar{x}}\right)\bar{y}\\ \bar{y}(0)=y_0/R,\ \bar{k}(0)=0\end{gathered} \tag{30}$$

This system contains, besides N, two parameters:

$$\varepsilon=y_0/R \text{ and } \bar{\mu}=\mu V(\rho_s h_f R)^{-1}$$

Once $\bar{y}(\bar{x})$ and $\bar{k}(x)$ have been determined from system (30), integration of the equation $\mu^{-1}dp/dx=-k/y^{*2}$ gives the pressure distribution, and integration of the components of the excess force of pressure and the frictional force in the direction of motion yield the resistance of the forward surface of the cylinder:

$$X=2\int_0^{\pi R/2}[\tau\cos\gamma+(p-p_d)\sin\gamma]\,dx=2\mu V\int_0^{\pi/2}\left[\frac{(\sin\bar{x}+\bar{k}/2)}{\bar{y}}\sin\bar{x}+\right.$$
$$\left.+\cos\bar{x}\int_0^{\pi/2}\frac{\bar{k}}{\bar{y}^2}\,d\bar{x}\right]d\bar{x}=2\mu V\bar{X}$$

Results of computations carried out with $\mu=0$ and $N=1$ and various values of the parameter $\varepsilon=\lambda(T_w-T_m)(\rho_s V h_f R)^{-1}$ are shown in Figs 14 and 15.

Figure 14 shows the shape of the melt layer surface; the dashed curve represents the surface of the cylinder, curves 1–4 correspond to values of $\varepsilon\times10^3=5$, 10, 25 and 50. Figure 15(a) shows plots of $\bar{k}$ illustrating the behaviour of the pressure gradient along the cylinder surface (with opposite sign); Figs 15(b and c) illustrate the distribution of excess pressure $p-p_d=\mu V(\varepsilon^2R)^{-1}\bar{p}$ and surface friction $\tau=\mu V(\varepsilon R)^{-1}\bar{\tau}$. Finally, the dots in Fig. 16 represent values of the resistance of the forward part of the cylinder, $X=2\mu V\varepsilon^{-2}\bar{X}(\varepsilon)$, and the total heat flux from it to the melting medium, $Q=2\rho_s Vh_j R\bar{Q}(\varepsilon)$.

The author is indebted to M. Slavinskii, who carried out the computations and prepared the figures.

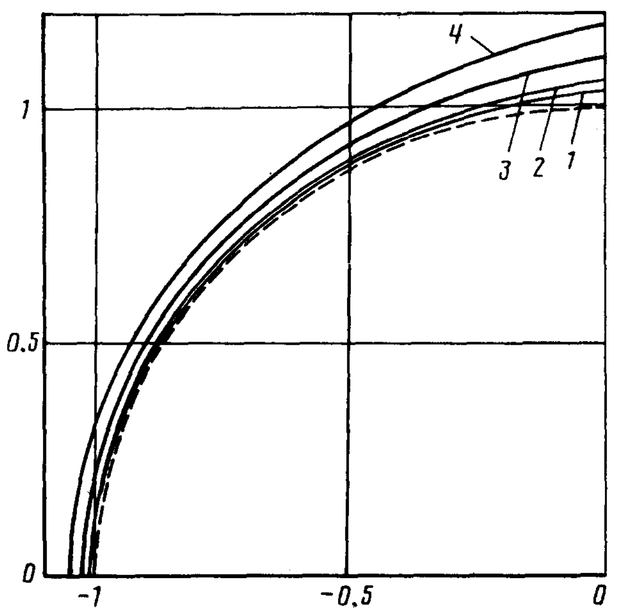

FIG. 14.

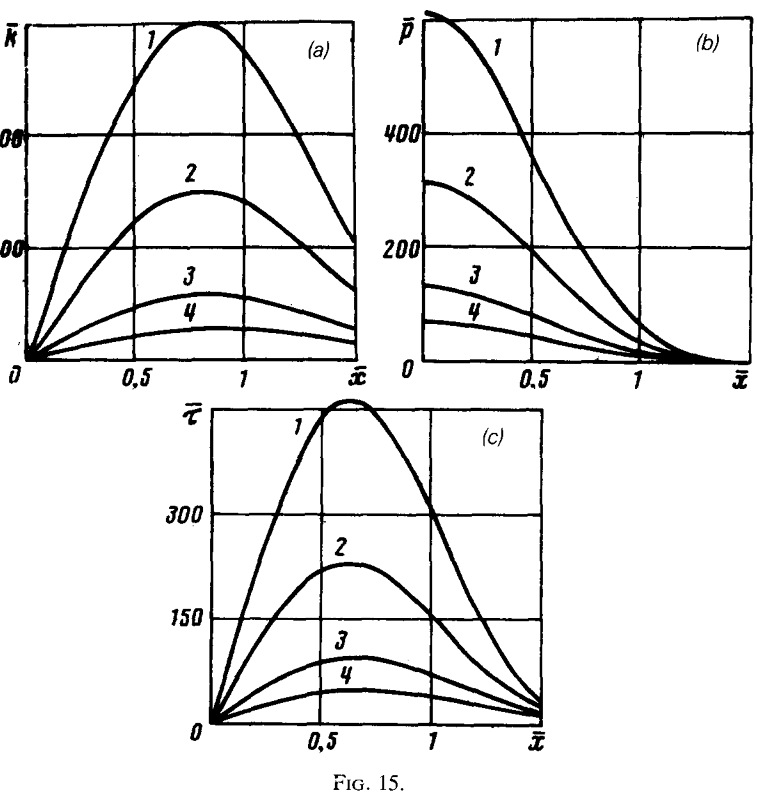

FIG. 15.

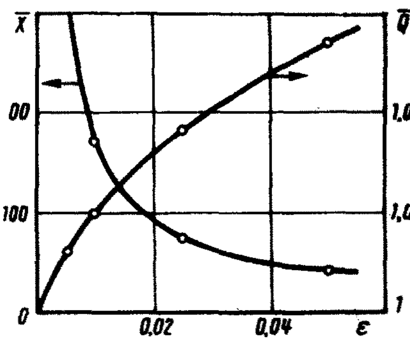

FIG. 16.

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Translated by D.L.